Exercise 2.4.2

Solve

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{with} \quad \begin{aligned} &\frac{\partial u}{\partial x}(0,t) = 0 \\ &u(L,t) = 0 \\ &u(x,0) = f(x). \end{aligned}$$

For this problem you may assume that no solutions of the heat equation exponentially grow in time. You may also guess appropriate orthogonality conditions for the eigenfunctions.

Solution

The heat equation and its associated boundary conditions are linear and homogeneous, so the method of separation of variables can be applied. Assume a product solution of the form $u(x,t) = X(x)T(t)$ and substitute it into the PDE

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \rightarrow \quad \frac{\partial}{\partial t}[X(x)T(t)] = k\frac{\partial^2}{\partial x^2}[X(x)T(t)]$$

and the boundary conditions.

$$\begin{aligned} \frac{\partial u}{\partial x}(0,t) = 0 \quad &\rightarrow \quad X'(0)T(t) = 0 \quad &\rightarrow \quad X'(0) = 0 \\ u(L,t) = 0 \quad &\rightarrow \quad X(L)T(t) = 0 \quad &\rightarrow \quad X(L) = 0 \end{aligned}$$

Now separate variables in the PDE.

$$X\frac{dT}{dt} = kT\frac{d^2X}{dx^2}$$

Divide both sides by $kX(x)T(t)$. Note that the final answer for u will be the same regardless which side k is on. Constants are normally grouped with t.

$$\underbrace{\frac{1}{kT}\frac{dT}{dt}}_{\text{function of } t} = \underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{\text{function of } x}$$

The only way a function of t can be equal to a function of x is if both are equal to a constant λ.

$$\frac{1}{kT}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = \lambda$$

As a result of applying the method of separation of variables, the PDE has reduced to two ODEs—one in x and one in t.

$$\left.\begin{aligned} \frac{1}{kT}\frac{dT}{dt} = \lambda \\ \frac{1}{X}\frac{d^2X}{dx^2} = \lambda \end{aligned}\right\}$$

Values of λ that result in nontrivial solutions for X and T are called the eigenvalues, and the solutions themselves are known as the eigenfunctions. Suppose first that λ is positive: $\lambda = \alpha^2$. The ODE for X becomes

$$\frac{d^2X}{dx^2} = \alpha^2 X.$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_1 \cosh \alpha x + C_2 \sinh \alpha x$$

Take a derivative with respect to x.

$$X'(x) = \alpha(C_1 \sinh \alpha x + C_2 \cosh \alpha x)$$

Apply the boundary conditions now to determine C_1 and C_2.

$$X'(0) = \alpha(C_2) = 0$$
$$X(L) = C_1 \cosh \alpha L + C_2 \sinh \alpha L = 0$$

The first equation implies that $C_2 = 0$, so the second equation reduces to $C_1 \cosh \alpha L = 0$. Because hyperbolic cosine is not oscillatory, C_1 must be zero for the equation to be satisfied. This results in the trivial solution $X(x) = 0$, which means there are no positive eigenvalues. Suppose secondly that λ is zero: $\lambda = 0$. The ODE for X becomes

$$\frac{d^2X}{dx^2} = 0.$$

The general solution is obtained by integrating both sides with respect to x twice.

$$\frac{dX}{dx} = C_3$$

Apply the boundary conditions at $x = 0$ now.

$$X'(0) = C_3 = 0$$

Consequently,

$$\frac{dX}{dx} = 0.$$

Integrate both sides with respect to x once more.

$$X(x) = C_4$$

Apply the boundary conditions at $x = L$ now.

$$X(L) = C_4 = 0$$

The trivial solution $X(x) = 0$ is obtained, so zero is not an eigenvalue. Suppose thirdly that λ is negative: $\lambda = -\beta^2$. The ODE for X becomes

$$\frac{d^2X}{dx^2} = -\beta^2 X.$$

The general solution is written in terms of sine and cosine.

$$X(x) = C_5 \cos \beta x + C_6 \sin \beta x$$

Take a derivative of it with respect to x.

$$X'(x) = \beta(-C_5 \sin \beta x + C_6 \cos \beta x)$$

Apply the boundary conditions now to determine C_5 and C_6.

$$X'(0) = \beta(C_6) = 0$$
$$X(L) = C_5 \cos \beta L + C_6 \sin \beta L = 0$$

The first equation implies that $C_6 = 0$, so the second equation reduces to $C_5 \cos \beta L = 0$. To avoid the trivial solution, we insist that $C_5 \neq 0$. Then

$$\begin{aligned} \cos \beta L &= 0 \\ \beta L &= \frac{1}{2}(2n-1)\pi, \quad n = 1, 2, \ldots \\ \beta_n &= \frac{1}{2L}(2n-1)\pi. \end{aligned}$$

There are negative eigenvalues $\lambda = -(2n-1)^2\pi^2/4L^2$, and the eigenfunctions associated with them are

$$\begin{aligned} X(x) &= C_5 \cos \beta x + C_6 \sin \beta x \\ &= C_5 \cos \beta x \quad \rightarrow \quad X_n(x) = \cos \frac{(2n-1)\pi x}{2L}. \end{aligned}$$

n only takes on the values it does because negative integers result in redundant values for λ. With this formula for λ, the ODE for T becomes

$$\frac{1}{kT}\frac{dT}{dt} = -\frac{(2n-1)^2\pi^2}{4L^2}.$$

Multiply both sides by kT.

$$\frac{dT}{dt} = -k\frac{(2n-1)^2\pi^2}{4L^2}T$$

The general solution is written in terms of the exponential function.

$$T(t) = C_7 \exp\left(-k\frac{(2n-1)^2\pi^2}{4L^2}t\right) \quad \rightarrow \quad T_n(t) = \exp\left(-\frac{k(2n-1)^2\pi^2}{4L^2}t\right)$$

According to the principle of superposition, the general solution to the PDE for u is a linear combination of $X_n(x)T_n(t)$ over all the eigenvalues.

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} A_n \exp\left(-\frac{k(2n-1)^2\pi^2}{4L^2}t\right) \cos \frac{(2n-1)\pi x}{2L}}$$

Now use the initial condition $u(x,0) = f(x)$ to determine A_n.

$$u(x,0) = \sum_{n=1}^{\infty} A_n \cos \frac{(2n-1)\pi x}{2L} = f(x)$$

Multiply both sides by $\cos[(2m-1)\pi x/2L]$, where m is an integer,

$$\sum_{n=1}^{\infty} A_n \cos \frac{(2n-1)\pi x}{2L} \cos \frac{(2m-1)\pi x}{2L} = f(x) \cos \frac{(2m-1)\pi x}{2L}$$

and then integrate both sides with respect to x from 0 to L.

$$\int_0^L \sum_{n=1}^{\infty} A_n \cos\frac{(2n-1)\pi x}{2L} \cos\frac{(2m-1)\pi x}{2L}\,dx = \int_0^L f(x)\cos\frac{(2m-1)\pi x}{2L}\,dx$$

Bring the constants in front of the integral on the left.

$$\sum_{n=1}^{\infty} A_n \int_0^L \cos\frac{(2n-1)\pi x}{2L} \cos\frac{(2m-1)\pi x}{2L}\,dx = \int_0^L f(x)\cos\frac{(2m-1)\pi x}{2L}\,dx$$

The cosine functions are orthogonal, so the integral on the left is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for the $n = m$ one.

$$A_n \int_0^L \cos^2\frac{(2n-1)\pi x}{2L}\,dx = \int_0^L f(x)\cos\frac{(2n-1)\pi x}{2L}\,dx$$

Evaluate the integral on the left side.

$$A_n\left(\frac{L}{2}\right) = \int_0^L f(x)\cos\frac{(2n-1)\pi x}{2L}\,dx$$

Therefore,

$$\boxed{A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{(2n-1)\pi x}{2L}\,dx.}$$